

Brans-Dicke Cosmology in $4D$ from scalar-vacuum in $5D$

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Abstract

We show that Brans-Dicke (BD) theory in $5D$ may explain the present cosmic accelerated expansion without recurring to matter fields in $5D$ or dark energy in $4D$. Without making any assumption on the nature of the extra coordinate or the matter content in $5D$, here we demonstrate that the vacuum BD field equations in $5D$ are equivalent, on every hypersurface orthogonal to the extra dimension, to a BD theory in $4D$ with a self interacting potential and an effective matter field. The potential is not introduced by hand, instead the reduction procedure provides an expression that determines its shape up to a constant of integration. It also establishes the explicit formulae for the effective matter in $4D$. In the context of FRW cosmologies, we show that the reduced BD theory gives rise to models for accelerated expansion of a matter-dominated universe which are consistent with current observations and with a decelerating radiation-dominated epoch.

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1 Introduction

One of the most challenging problems in cosmology today is to explain the observed late-time accelerated expansion of the universe [1]-[13]. Since the gravity of both baryonic (ordinary) matter and radiation is attractive, an accelerated expansion requires the presence of a new form of matter (called *dark energy*), which could (i) produce gravitational repulsion, i.e., violate the strong energy condition; (ii) account for 70% of the total content of the universe; (iii) remain unclustered on all scales where gravitational clustering of ordinary matter is seen. (For a recent review see Ref. [14]). Possible candidates for dark energy include: a cosmological constant or a time dependent cosmological term [15]-[16]; an evolving scalar field known as *quintessence* (Q -matter) with a potential giving rise to negative pressure at the present epoch [17]-[20]; dissipative fluids [21]; Chaplygin gas [22]-[23]; K-essence [24]-[27], and other more exotic models [28].

General Relativity (GR) is a well tested theory on solar system scales. In contrast it is poorly tested on cosmic scales [29]. Therefore, the question arises of whether dark energy is not just an observational artifact caused by an inappropriate theory of gravity. This and other puzzles of theoretical and experimental gravity have triggered a huge interest in alternative theories of gravity (See, e.g. [30] and references therein). In recent years there has been a renewed interest in scalar-tensor theories of gravity as viable alternatives to general relativity. In particular, some researchers [31]-[37] have resorted to Brans-Dicke theory (BD) [38] in order to explain the present accelerated expansion of the universe. An attractive feature of BD is that the scalar field is a fundamental element of the theory, as opposed to other models in which the scalar field is postulated separately in an ad hoc fashion. The concept is that the BD scalar field could play the role of Q -matter or K -essence and lead to cosmological acceleration. However, it turns out that this is so only in very particular cases: for values of the coupling parameter ω in the range $-2 < \omega < -3/2$, which not only violate the energy condition on the scalar field but are also inconsistent with a radiation-dominated epoch, unless ω varies with time [32]; when there is a scalar potential, which in turn is added by hand [33]. Further models include the so-called “chameleon fields” that allow the scalar field to interact with matter (see [39] and references therein).

Another attempt to derive the present accelerated expansion from a fundamental concept (or postulate) combines the original BD and Kaluza-Klein (KK) theories with the modern view that our $4D$ universe can be recovered on a hypersurface orthogonal to the extra dimension. Along these lines, and assuming that the extra coordinate is cyclic (ignorable), it has been shown in [40] that the BD field equations in $5D$ are equivalent to those of GR in $4D$ with two scalar fields, viz., the BD scalar field and the γ_{55} component of the KK metric (For a recent discussion see Ref. [41]). These two scalar fields may account for the present accelerated expansion of the universe, if one admits the existence of matter in $5D$ which does not move along the extra dimension and effectively behaves as dust in $4D$.

In this work we adhere to the point of view advanced in [40]-[41]. However, we propose that neither the cylindricity condition on the extra coordinate nor the higher dimensional matter hypothesis are necessary. In fact, in this paper we show that, regardless of whether we assume a compact or large extra dimension, BD theory in $5D$ may explain the present accelerated expansion without introducing matter fields in $5D$ or dark energy in $4D$. We demonstrate that the vacuum BD field equations in $5D$ are equivalent, on every hypersurface orthogonal to the extra dimension, to a BD theory in $4D$ with a self interacting potential and an effective matter field. In the context of FRW cosmologies, we show that the reduced BD theory gives rise to models for accelerated expansion of a matter-dominated universe which are consistent with a decelerating radiation-dominated epoch, for the same value of the BD parameter ω .

The paper is organized as follows. In section 2 we perform the dimensional reduction of the scalar-vacuum BD field equations in $5D$. Our procedure provides explicit definitions for the effective matter and potential¹ in $4D$. In section 3 we study homogeneous and isotropic solutions of the vacuum BD field equations in $5D$ under the assumption of separation of variables (in general, the metric functions may depend on time and the extra coordinate). We show that there are four classes of separable solutions. We discuss in detail the class of power-law solutions, which in turn consists of five families of solutions. In section 4 we study the $4D$ cosmological scenarios allowed by the $5D$ power-law solutions derived in section 3. We find that distinct families of solutions in $5D$ lead to very much alike scenarios in $4D$; they can give accelerating matter-dominated era as well as decelerating radiation-dominated era for the same value of ω in the range $-3/2 < \omega < -1$. In section 5 we give a summary of our results. Finally, in the

¹A similar procedure has recently been discussed in [42], where a “modified” BD theory in $4D$ is obtained from the vacuum BD equations in $5D$. However, our formulae and definitions of the $4D$ quantities are completely different from those derived in [42].

Appendix we show, by means of explicit integration, that the assumption of separability is consistent with the field equations.

2 Dimensional reduction of Brans-Dicke theory in 5D

The Brans-Dicke (BD) theory of gravity in 5D is described by the action [40]

$$S_{(5)} = \int d^5x \sqrt{|\gamma^{(5)}|} \left[\phi R^{(5)} - \frac{\omega}{\phi} \gamma^{AB} (\nabla_A \phi) (\nabla_B \phi) \right] + 16\pi \int d^5x \sqrt{|\gamma^{(5)}|} L_m^{(5)}, \quad (1)$$

where $R^{(5)}$ is the curvature scalar associated with the 5D metric γ_{AB} ; $\gamma^{(5)}$ is the determinant of γ_{AB} ; ϕ is a scalar field; ω is a dimensionless coupling constant; $L_m^{(5)}$ represents the Lagrangian of the matter fields in 5D and does not depend on ϕ .

The equations for the gravitational field in 5D derived from (1) read

$$G_{AB}^{(5)} = R_{AB}^{(5)} - \frac{1}{2} \gamma_{AB} R^{(5)} = \frac{8\pi}{\phi} T_{AB}^{(5)} + \frac{\omega}{\phi^2} \left[(\nabla_A \phi) (\nabla_B \phi) - \frac{1}{2} \gamma_{AB} (\nabla^C \phi) (\nabla_C \phi) \right] + \frac{1}{\phi} (\nabla_A \nabla_B \phi - \gamma_{AB} \nabla^2 \phi), \quad (2)$$

where $\nabla^2 \equiv \nabla_A \nabla^A$ and $T_{AB}^{(5)}$ represents the energy-momentum tensor (EMT) of matter fields in 5D with trace $T^{(5)} = \gamma^{AB} T_{AB}^{(5)}$. For the sake of generality of the reduction procedure in this section $T_{AB}^{(5)} \neq 0$.

The field equation for the scalar field ϕ is determined by (1) as

$$\frac{2\omega}{\phi} \nabla^2 \phi - \frac{\omega}{\phi^2} (\nabla_A \phi) (\nabla^A \phi) + R^{(5)} = 0. \quad (3)$$

Taking the trace of (2) we find

$$R^{(5)} = -\frac{16\pi}{3\phi} T^{(5)} + \frac{\omega}{\phi^2} (\nabla_A \phi) (\nabla^A \phi) + \frac{8}{3\phi} \nabla^2 \phi. \quad (4)$$

Combining the last two equations we get

$$\nabla^2 \phi = \frac{8\pi}{4 + 3\omega} T^{(5)}. \quad (5)$$

In this work we use coordinates where the metric in 5D can be written as²

$$dS^2 = \gamma_{AB} dx^A dx^B = g_{\mu\nu}(x, y) dx^\mu dx^\nu + \epsilon \Phi^2(x, y) dy^2, \quad (6)$$

in such a way that our 4D spacetime can be recovered by going onto a hypersurface $\Sigma_y : y = y_0 = \text{constant}$, which is orthogonal to the 5D unit vector

$$\hat{n}^A = \frac{\delta_4^A}{\Phi}, \quad n_A n^A = \epsilon, \quad (7)$$

along the extra dimension, and $g_{\mu\nu}$ can be interpreted as the metric of the spacetime.

The effective field equations (FE) in 4D are obtained from dimensional reduction of (2) and (5). To achieve such a reduction we note that

$$\begin{aligned} \nabla_\mu \nabla_\nu \phi &= D_\mu D_\nu \phi + \frac{\epsilon}{2\Phi^2} g_{\mu\nu}^* \phi^*, \\ \nabla_4 \nabla_4 \phi &= \epsilon \Phi (D_\alpha \Phi) (D^\alpha \phi) + \phi^{**} - \frac{\Phi^*}{\Phi} \phi^*, \\ \nabla^2 \phi &= D^2 \phi + \frac{(D_\alpha \Phi) (D^\alpha \phi)}{\Phi} + \frac{\epsilon}{\Phi^2} \left[\phi^{**} + \phi^* \left(\frac{g^{\mu\nu} g_{\mu\nu}^*}{2} - \frac{\Phi^*}{\Phi} \right) \right] \end{aligned} \quad (8)$$

²Notation: $x^\mu = (x^0, x^1, x^2, x^3)$ are the coordinates in 4D and y is the coordinate along the extra dimension. We use spacetime signature $(+, -, -, -)$, while $\epsilon = \pm 1$ allows for spacelike or timelike extra dimension.

where the asterisk denotes partial derivative with respect to the extra coordinate (i.e., $\partial/\partial y = *$); D_α is the covariant derivative on Σ_y , which is calculated with $g_{\mu\nu}$, and $D^2 \equiv D^\alpha D_\alpha$.

Using these expressions, the spacetime components ($A = \mu, B = \nu$) of the 5D field equations (2) can be written as

$$G_{\mu\nu}^{(5)} = \frac{8\pi}{\phi} T_{\mu\nu}^{(5)} + \frac{\omega}{\phi^2} \left[(D_\mu \phi) (D_\nu \phi) - \frac{1}{2} g_{\mu\nu} (D_\alpha \phi) (D^\alpha \phi) \right] + \frac{1}{\phi} (D_\mu D_\nu \phi - g_{\mu\nu} D^2 \phi) - \frac{g_{\mu\nu} (D_\alpha \Phi) (D^\alpha \phi)}{\Phi \phi} - \frac{\epsilon g_{\mu\nu}}{2\Phi^2 \phi} \left[2 \frac{**}{\phi} + \frac{*}{\phi} \left(g^{\alpha\beta} g_{\alpha\beta}^* - 2 \frac{\Phi}{\Phi} + \omega \frac{\phi}{\phi} \right) \right] + \frac{\epsilon g_{\mu\nu}^* \phi}{2\Phi^2 \phi}. \quad (9)$$

To construct the Einstein tensor in 4D we have to express $R_{\alpha\beta}^{(5)}$ and $R^{(5)}$ in terms of the corresponding 4D quantities. The Ricci tensor $R_{\mu\nu}^{(4)}$ of the metric $g_{\mu\nu}$ and the scalar field Φ are related to the Ricci tensor $R_{AB}^{(5)}$ of γ_{AB} by [43]

$$R_{\alpha\beta}^{(5)} = R_{\alpha\beta}^{(4)} - \frac{D_\alpha D_\beta \Phi}{\Phi} + \frac{\epsilon}{2\Phi^2} \left(\frac{\Phi g_{\alpha\beta}^*}{\Phi} - g_{\alpha\beta}^{**} + g^{\lambda\mu} g_{\alpha\lambda}^* g_{\beta\mu}^* - \frac{g^{\mu\nu} g_{\mu\nu}^* g_{\alpha\beta}^*}{2} \right),$$

$$R_{44}^{(5)} = -\epsilon \Phi D^2 \Phi - \frac{g^{\lambda\beta} g_{\lambda\beta}^*}{4} - \frac{g^{\lambda\beta} g_{\lambda\beta}^{**}}{2} + \frac{\Phi g^{\lambda\beta} g_{\lambda\beta}^*}{2\Phi}. \quad (10)$$

From (2)-(4) and the second equation in (10) we obtain

$$\frac{D^2 \Phi}{\Phi} = - \frac{(D_\alpha \Phi) (D^\alpha \phi)}{\Phi \phi} - \frac{\epsilon}{2\Phi^2} \left[g^{\lambda\beta} g_{\lambda\beta}^{**} + \frac{g^{\lambda\beta} g_{\lambda\beta}^*}{2} - \frac{\Phi g^{\lambda\beta} g_{\lambda\beta}^*}{\Phi} \right] - \frac{\epsilon}{\Phi^2 \phi} \left[\frac{**}{\phi} + \frac{*}{\phi} \left(\frac{\omega \phi^*}{\phi} - \frac{\Phi}{\Phi} \right) \right]$$

$$+ \frac{8\pi}{\phi} \left[\frac{(\omega + 1) T^{(5)}}{4 + 3\omega} - \frac{\epsilon T_{44}^{(5)}}{\Phi^2} \right]. \quad (11)$$

Substituting this expression into $R^{(5)} = \gamma^{AB} R_{AB}$ we find

$$R^{(5)} = R^{(4)} + \frac{2 (D_\alpha \Phi) (D^\alpha \phi)}{\Phi \phi} - \frac{\epsilon}{4\Phi^2} \left[g^{\alpha\beta} g_{\alpha\beta}^* + \left(g^{\alpha\beta} g_{\alpha\beta}^* \right)^2 \right] + \frac{2\epsilon}{\Phi^2 \phi} \left[\frac{**}{\phi} + \frac{*}{\phi} \left(\frac{\omega \phi^*}{\phi} - \frac{\Phi}{\Phi} \right) \right]$$

$$+ \frac{16\pi}{\phi} \left[\frac{\epsilon T_{44}^{(5)}}{\Phi^2} - \frac{(\omega + 1) T^{(5)}}{4 + 3\omega} \right], \quad (12)$$

where $R^{(4)} = g^{\alpha\beta} R_{\alpha\beta}^{(4)}$ is the scalar curvature of the spacetime hypersurfaces Σ_y .

• We are now ready to obtain the effective equations for gravity in 4D. With this aim we substitute the first equation in (10) and (12) into (9) and isolate $G_{\mu\nu}^{(4)} = R_{\mu\nu}^{(4)} - g_{\mu\nu} R^{(4)}/2$. The result can be written as

$$G_{\mu\nu}^{(4)} = \frac{8\pi}{\phi} (S_{\mu\nu} + T_{\mu\nu}^{(BD)}) + \frac{\omega}{\phi^2} \left[(D_\mu \phi) (D_\nu \phi) - \frac{1}{2} g_{\mu\nu} (D_\alpha \phi) (D^\alpha \phi) \right] + \frac{1}{\phi} (D_\mu D_\nu \phi - g_{\mu\nu} D^2 \phi) - g_{\mu\nu} \frac{V(\phi)}{2\phi} \quad (13)$$

where we have introduced the quantity $V(\phi)$, which (as we will see bellow) plays the role of an effective or induced scalar potential; $S_{\mu\nu}$ is the reduced EMT of the matter fields in 5D

$$S_{\mu\nu} = T_{\mu\nu}^{(5)} - g_{\mu\nu} \left[\frac{(\omega + 1) T^{(5)}}{4 + 3\omega} - \frac{\epsilon T_{44}^{(5)}}{\Phi^2} \right], \quad (14)$$

and $T_{\mu\nu}^{(BD)}$ can be interpreted as an induced EMT for an effective BD theory in $4D$. It is given by

$$8\pi T_{\mu\nu}^{(BD)} = 8\pi T_{\mu\nu}^{(STM)} + \frac{\epsilon}{2\Phi^2} \left[g_{\mu\nu}^* + g_{\mu\nu} \left(\frac{\omega}{\phi} \frac{\phi^*}{\phi} - g^{\alpha\beta} g_{\alpha\beta}^* \right) \right] + \frac{1}{2} g_{\mu\nu} V, \quad (15)$$

with

$$\frac{8\pi}{\phi} T_{\mu\nu}^{(STM)} \equiv \frac{D_\mu D_\nu \Phi}{\Phi} - \frac{\epsilon}{2\Phi^2} \left\{ \frac{\Phi g_{\mu\nu}^*}{\Phi} - g_{\mu\nu}^{**} + g^{\alpha\beta} g_{\mu\alpha}^* g_{\nu\beta}^* - \frac{g^{\alpha\beta} g_{\alpha\beta}^* g_{\mu\nu}^*}{2} + \frac{g_{\mu\nu}}{4} \left[g^{*\alpha\beta} g_{\alpha\beta}^* + \left(g^{\alpha\beta} g_{\alpha\beta}^* \right)^2 \right] \right\} \quad (16)$$

Since in BD ϕ acts as the inverse of the Newtonian gravitational constant G , (16) is identical to the induced EMT used in STM (Space-Time-Matter theory) [43]. The second term in (15) depends on the first derivatives of ϕ with respect to the fifth coordinate and represents the effective EMT in $4D$ coming from the scalar field. The equation for the scalar potential V is given bellow by (19).

Taking the trace of (13) we obtain a simple relation between $R^{(4)}$, $S = g^{\mu\nu} S_{\mu\nu}$ and $T^{(BD)} = g^{\mu\nu} T_{\mu\nu}^{(BD)}$, namely (we note that $g^{\mu\nu} T_{\mu\nu}^{(5)} = T^{(5)} - \epsilon T_{44}^{(4)} / \Phi^2$)

$$R^{(4)} = -\frac{8\pi}{\phi} (S + T^{(BD)}) + \frac{\omega (D_\alpha \phi) (D^\alpha \phi)}{\phi^2} + \frac{3D^2 \phi}{\phi} + \frac{2V}{\phi}. \quad (17)$$

- To construct the $4D$ counterpart of (5) we substitute (12) and (17) into (3). After some manipulations we get

$$D^2 \phi = \frac{8\pi}{3+2\omega} (S + T^{(BD)}) + \frac{1}{3+2\omega} \left[\phi \frac{dV(\phi)}{d\phi} - 2V(\phi) \right], \quad (18)$$

where

$$\begin{aligned} \phi \frac{dV(\phi)}{d\phi} \equiv & -2(1+\omega) \left[\frac{(D_\alpha \Phi) (D^\alpha \phi)}{\Phi} + \frac{\epsilon}{\Phi^2} \left(\phi^{**} - \frac{\Phi \phi^*}{\Phi} \right) \right] - \frac{\epsilon \omega}{\Phi^2} \left(\frac{\phi^*}{\phi} + g^{\mu\nu} g_{\mu\nu}^* \right) \\ & + \frac{\epsilon \phi}{4\Phi^2} \left[g^{*\alpha\beta} g_{\alpha\beta}^* + \left(g^{\alpha\beta} g_{\alpha\beta}^* \right)^2 \right] + 16\pi \left[\frac{(1+\omega) T^{(5)}}{4+3\omega} - \frac{\epsilon T_{44}^{(5)}}{\Phi^2} \right]. \end{aligned} \quad (19)$$

This equation, with the r.h.s. evaluated at some Σ_y , constitutes a working definition for the potential.

We notice that equations (13) and (18) are identical to those of Brans-Dicke theory in $4D$ derived from the action

$$S_{(4)} = \int d^4x \sqrt{-g} \left[\phi R^{(4)} - \frac{\omega}{\phi} g^{\mu\nu} (D_\mu \phi) (D_\nu \phi) - V(\phi) \right] + 16\pi \int d^4x \sqrt{-g} L_m^{(4)}, \quad (20)$$

with $\sqrt{-g} [S_{\mu\nu} + T_{\mu\nu}^{(BD)}] \equiv 2\delta S_m / \delta g^{\mu\nu}$, where $S_m = \int d^4x \sqrt{-g} L_m^{(4)}$ represents the action for matter in $4D$.

The potential vanishes only in few particular cases. For example when $\omega = -1$, y is a cyclic coordinate and $T_{AB}^{(5)} = 0$. However, if we are to recover a general version of BD theory in $4D$, we must assume $\omega \neq -1$. Besides, the metric in $5D$ can depend on y . Therefore, in general $V \neq 0$.

- Finally, for the line element (6) we have $R_{4\alpha}^{(5)} = \Phi P_{\alpha;\beta}^{(5)}$ [43] with

$$P_{\alpha\beta} = \frac{1}{2\Phi} \left(g_{\alpha\beta}^* - g_{\alpha\beta} g^{\mu\nu} g_{\mu\nu}^* \right). \quad (21)$$

The dynamical equation for $P_{\alpha\beta}$ is obtained from (2) by setting $A = 4$, $B = \mu$, viz.,

$$(\Phi \phi) P_{\mu;\alpha}^\alpha = 8\pi T_{\mu 4}^{(5)} + \frac{\omega}{\phi} \frac{\phi^* D_\mu \phi}{\phi} + D_\mu \frac{\phi^*}{\phi} - \frac{1}{2} g_{\mu\lambda}^* D^\lambda \phi - \frac{\phi^* D_\mu \Phi}{\Phi}. \quad (22)$$

In the case where $T_{\mu^4}^{(5)} = 0$ and $\phi = \text{constant}$, this reduces to $P_{\mu;\alpha}^\alpha = 0$. In braneworld theory this quantity is proportional to the EMT of the matter on the brane [44]. If y is a cyclic coordinate, (22) reduces to $0 = 0$.

To keep contact with other works in the literature, we note that we can substitute (11) in (14) to obtain an expression where the right hand side of (13) only contains $T_{\mu\nu}^{(5)}$ as well as the derivatives of ϕ , Φ and $g_{\mu\nu}$. In case that there is no y dependence we neatly recover the formulae developed in [40]. However, our equations are completely different from those derived/used in [42].

3 Scalar-vacuum Brans-Dicke cosmology in 5D

In cosmological applications the 5D metric (6) is commonly taken in the form

$$dS^2 = n^2(t, y)dt^2 - a^2(t, y) \left[\frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right] + \epsilon \Phi^2(t, y) dy^2, \quad (23)$$

where $k = 0, +1, -1$ and (t, r, θ, ϕ) are the usual coordinates for a spacetime with spherically symmetric spatial sections³.

For this line element the vacuum ($T_{AB}^{(5)} = 0$) Brans-Dicke field equations (2) reduce as follows: The temporal component $A = B = 0$ gives

$$3 \frac{\dot{a}}{a} \left(\frac{\dot{a}}{a} + \frac{\dot{\Phi}}{\Phi} \right) + \frac{3kn^2}{a^2} + \frac{3\epsilon n^2}{\Phi^2} \left[\frac{a^{**}}{a} + \frac{a^*}{a} \left(\frac{a^*}{a} - \frac{\Phi^*}{\Phi} \right) \right] = \frac{1}{\phi} \left[\ddot{\phi} + \dot{\phi} \left(\frac{\omega \dot{\phi}}{2\phi} - \frac{\dot{n}}{n} \right) \right] + \frac{\epsilon n^2}{\Phi^2 \phi} \left(\frac{n^*}{n} - \frac{\omega \phi^*}{2\phi} \right); \quad (24)$$

the spatial components $A = B = 1, 2, 3$ reduce to

$$\begin{aligned} \frac{2\ddot{a}}{a} + \frac{\dot{a}}{a} \left(\frac{\dot{a}}{a} - \frac{2\dot{n}}{n} \right) + \frac{\ddot{\Phi}}{\Phi} + \frac{\dot{\Phi}}{\Phi} \left(\frac{2\dot{a}}{a} - \frac{\dot{n}}{n} \right) + \frac{kn^2}{a^2} + \frac{\epsilon n^2}{\Phi^2} \left[\frac{2a^{**}}{a} + \frac{a^*}{a} \left(\frac{a^*}{a} + \frac{2n^*}{n} \right) + \frac{n^{**}}{n} - \frac{\Phi^*}{\Phi} \left(\frac{2a^*}{a} + \frac{n^*}{n} \right) \right] \\ = \frac{\dot{\phi}}{\phi} \left(\frac{\dot{a}}{a} - \frac{\omega \dot{\phi}}{2\phi} \right) + \frac{\epsilon n^2}{\Phi^2 \phi} \left(\frac{a^*}{a} - \frac{\omega \phi^*}{2\phi} \right); \end{aligned} \quad (25)$$

the $A = B = 4$ component gives

$$3 \left[\frac{\ddot{a}}{a} + \frac{\dot{a}}{a} \left(\frac{\dot{a}}{a} - \frac{\dot{n}}{n} \right) \right] + \frac{3kn^2}{a^2} + \frac{3\epsilon n^2}{\Phi^2} \left(\frac{a^*}{a} \right) \left(\frac{a^*}{a} + \frac{n^*}{n} \right) = \frac{\dot{\phi}}{\phi} \left(\frac{\dot{\Phi}}{\Phi} - \frac{\omega \dot{\phi}}{2\phi} \right) + \frac{\epsilon n^2}{\Phi^2 \phi} \left[\phi^{**} + \phi^* \left(\frac{\omega \phi^*}{2\phi} - \frac{\Phi^*}{\Phi} \right) \right]; \quad (26)$$

the mixed component $A = 0, B = 4$ yields

$$3 \left(\frac{n^* \dot{a}}{na} + \frac{\dot{\Phi} a^*}{\Phi a} - \frac{a^*}{a} \right) = \frac{\dot{\phi}}{\phi} - \frac{n^* \dot{\phi}}{n\phi} - \frac{\dot{\Phi} \phi^*}{\Phi \phi} + \frac{\omega}{\phi^2} \dot{\phi} \phi^*. \quad (27)$$

Finally, the wave equation (5) becomes

$$\nabla^2 \phi = \frac{1}{n^2} \left[\ddot{\phi} + \dot{\phi} \left(\frac{3\dot{a}}{a} + \frac{\dot{\Phi}}{\Phi} - \frac{\dot{n}}{n} \right) \right] + \frac{\epsilon}{\Phi^2} \left[\phi^{**} + \phi^* \left(\frac{3a^*}{a} + \frac{n^*}{n} - \frac{\Phi^*}{\Phi} \right) \right] = 0. \quad (28)$$

³We do not make any assumption about ϵ . Rather we let the FE to determine the signature of the extra dimension. In Cases 1 and 5 (with the exception of solution (68)) the extra dimension must be spacelike. All other solutions allow both signatures.

3.1 Separation of variables

In this section, we look for solutions to the above equations under the assumption that the metric coefficients are separable functions of their arguments. In this framework, without loss of generality we can set⁴

$$n(t, y) = N(y), \quad a(t, y) = P(y)Q(t), \quad \Phi(t, y) = F(t), \quad \phi(t, y) = U(y)W(t). \quad (29)$$

Then, from (27) we obtain

$$\frac{3\dot{Q}}{Q} \left(\frac{\dot{N}}{N} - \frac{\dot{P}}{P} \right) + \frac{\dot{F}}{F} \left(\frac{3\dot{P}}{P} + \frac{\dot{U}}{U} \right) + \frac{\dot{W}}{W} \left(\frac{\dot{N}}{N} - \frac{(1+\omega)\dot{U}}{U} \right) = 0. \quad (30)$$

Thus, separability requires

$$\frac{\dot{N}}{N} - \frac{\dot{P}}{P} = c_1 \frac{\dot{f}}{f}, \quad \frac{3\dot{P}}{P} + \frac{\dot{U}}{U} = c_2 \frac{\dot{f}}{f}, \quad \frac{\dot{N}}{N} - \frac{(1+\omega)\dot{U}}{U} = c_3 \frac{\dot{f}}{f}, \quad (31)$$

where c_1, c_2, c_3 are arbitrary constants and $f = f(y)$ is some function to be determined by the field equations.

From (31) we get

$$N = N_0 f^{[(3c_1+c_2)(\omega+1)+c_3]/(3\omega+4)}, \quad P = P_0 f^{[c_2(1+\omega)-c_1+c_3]/(3\omega+4)}, \quad U = U_0 f^{[c_2+3(c_1-c_3)]/(3\omega+4)}, \quad (32)$$

where N_0, P_0 and U_0 are constants of integration. Besides, from (30) and (31) it follows that

$$Q^{3c_1} F^{c_2} W^{c_3} = \text{constant}. \quad (33)$$

Now, to obtain an equation for f we substitute (29) and (32) into the wave equation (28). The requirement of separability yields a second-order differential equation for $f(y)$ whose first integral is

$$\left(\frac{df}{dy} \right)^2 = (C_s f^{2c_2} + C) f^{2[(3-3c_1-4c_2)\omega-3c_1-5c_2-c_3+4]/(3\omega+4)}, \quad (34)$$

where C_s is a separation constant and C is a constant of integration. This equation admits exact integration, in terms of elementary functions, in several cases. For example, setting $C = 0$ we get

$$f(y) \propto \begin{cases} y^{(3\omega+4)/[(3c_1+c_2)(\omega+1)+c_3]}, & (3c_1+c_2)(\omega+1)+c_3 \neq 0, \\ e^{\pm\sqrt{C_s}y}, & (3c_1+c_2)(\omega+1)+c_3 = 0, \quad C_s > 0. \end{cases} \quad (35)$$

Similar power-law and exponential solutions can be obtained for $C_s = 0$ or $c_2 = 0$.

Next, we substitute the metric functions (29), with (32) and (34), into the FE (24)-(26). We find that these equations have the following structure

$$H_1(t) + \eta_0 C H_2(t) [f(y)]^{-2c_2} + k(3\omega+4) H_3(t) [f(y)]^{2c_1} = 0, \quad (36)$$

where $\eta_0 \equiv c_2^2(5+4\omega)+2c_2[c_3+3c_1(1+\omega)]-3(c_3-c_1)^2$; $H_1(t)$ is a combination of $F(t)$, $Q(t)$ and $W(t)$ and their derivatives; $H_2 \propto W^2 Q^2$; $H_3 \propto W^2 F^2$. Inspection of the FE shows that the last two terms in (36) do not cancel out for $c_2 = -c_1 \neq 0$ and $F \propto Q$. Therefore, the assumption of separability demands either $c_1 = 0$ for $k \neq 0$, or $k = 0$ for $c_1 \neq 0$ as well as the fulfillment of (at least) one of the following conditions: $C = 0$, $\eta_0 = 0$, $c_2 = 0$.

A detailed analysis of the field equations is provided in the Appendix, where we derive the differential equations to be satisfied by the three functions Q , F and W . Inspection of those equations reveals that they admit power-law and exponential solutions for several choices of the parameters. However, this is clear at the outset if we notice that (30) can also be written as

⁴This is a consequence of the freedom to perform the coordinate transformation $t = t(\bar{t})$, $y = y(\bar{y})$.

$$\frac{3}{P} \frac{\dot{P}}{P} \left(\frac{\dot{F}}{F} - \frac{\dot{Q}}{Q} \right) + \frac{\dot{N}}{N} \left(\frac{3\dot{Q}}{Q} + \frac{\dot{W}}{W} \right) + \frac{\dot{U}}{U} \left(\frac{\dot{F}}{F} - \frac{(1+\omega)\dot{W}}{W} \right) = 0, \quad (37)$$

which requires

$$\frac{\dot{F}}{F} - \frac{\dot{Q}}{Q} = \bar{c}_1 \frac{\dot{h}}{h}, \quad \frac{3\dot{Q}}{Q} + \frac{\dot{W}}{W} = \bar{c}_2 \frac{\dot{h}}{h}, \quad \frac{\dot{F}}{F} - \frac{(1+\omega)\dot{W}}{W} = \bar{c}_3 \frac{\dot{h}}{h}, \quad (38)$$

where $\bar{c}_1, \bar{c}_2, \bar{c}_3$ are arbitrary constants and $h = h(t)$ is some function to be determined by the field equations. Integrating (38) we get

$$F \propto h^{[(3\bar{c}_1 + \bar{c}_2)(\omega+1) + \bar{c}_3]/(3\omega+4)}, \quad Q \propto h^{[\bar{c}_2(\omega+1) - \bar{c}_1 + \bar{c}_3]/(3\omega+4)}, \quad W \propto h^{[\bar{c}_2 + 3(\bar{c}_1 - \bar{c}_3)]/(3\omega+4)} \quad (39)$$

Following the same procedure as above, from the wave equation (28) we find

$$\left(\frac{dh}{dt} \right)^2 = (\bar{C}_s h^{2\bar{c}_2} + \bar{C}) h^{2[(3-3\bar{c}_1-4\bar{c}_2)\omega - 3\bar{c}_1 - 5\bar{c}_2 - \bar{c}_3 + 4]/(3\omega+4)}, \quad (40)$$

where \bar{C}_s and \bar{C} are some new separation and integration constants, respectively. For several choices of the constants we obtain power-law and exponential solutions. For example, if we set $\bar{C} = 0$ we get

$$h(t) \propto \begin{cases} t^{(3\omega+4)/[(3\bar{c}_1 + \bar{c}_2)(\omega+1) + \bar{c}_3]}, & (3\bar{c}_1 + \bar{c}_2)(\omega+1) + \bar{c}_3 \neq 0, \\ e^{\pm \sqrt{\bar{C}_s} t}, & (3\bar{c}_1 + \bar{c}_2)(\omega+1) + \bar{c}_3 = 0, \quad \bar{C}_s > 0. \end{cases} \quad (41)$$

Once again, similar solutions are obtained $\bar{C}_s = 0$ or $\bar{c}_2 = 0$.

In summary, the assumption of separation of variables generates four distinct classes of solutions corresponding to the following choices:

$$\begin{aligned} \text{I} &: h(t) \propto t^{(3\omega+4)/[(3\bar{c}_1 + \bar{c}_2)(\omega+1) + \bar{c}_3]}, \quad f(y) \propto y^{(3\omega+4)/[(3c_1 + c_2)(\omega+1) + c_3]}, \\ \text{II} &: h(t) \propto e^{\pm \sqrt{\bar{C}_s} t}, \quad f(y) \propto y^{(3\omega+4)/[(3c_1 + c_2)(\omega+1) + c_3]}, \\ \text{III} &: h(t) \propto t^{(3\omega+4)/[(3\bar{c}_1 + \bar{c}_2)(\omega+1) + \bar{c}_3]}, \quad f(y) \propto e^{\pm \sqrt{\bar{C}_s} y}, \\ \text{IV} &: h(t) \propto e^{\pm \sqrt{\bar{C}_s} t}, \quad f(y) \propto e^{\pm \sqrt{\bar{C}_s} y}, \end{aligned} \quad (42)$$

where the constants obey the conditions indicated in (35) and (41). When we substitute (42) in the FE (24)-(28) the latter reduce to a system of algebraic equations which provide the appropriate consistency relations for the constants.

3.2 Power-law solutions

We now proceed to study in some detail the power-law solutions. We concentrate our attention on the spatially-flat scenario, which seem to be relevant to the present epoch of the universe [45].

We could use the analysis provided in the Appendix and study case by case the differential equations for Q, F and W . However, since we already know the form of the desired solutions it is much easier to start from the family I in (42). For practical reasons it is convenient to adopt another parameterization, namely $(c_1, c_2, c_3) \rightarrow (\alpha, \beta, \gamma)$ and $(\bar{c}_1, \bar{c}_2, \bar{c}_3) \rightarrow (l, m, s)$, where the power-law solution takes the form

$$n = Ay^\alpha, \quad a = By^{\beta} t^l, \quad \Phi = Ct^m, \quad \phi = Dy^\gamma t^s. \quad (43)$$

Here A, B, C, D are some constants with the appropriate units; while α, β, γ and l, m, s are parameters that have to satisfy the field equations (24)-(28).

Substituting (43) into (28) we obtain

$$s(s-1+3l+m)C^2 t^{(s-2)} y^{(\gamma-2\alpha)} + \epsilon \gamma (\gamma-1+3\beta+\alpha) A^2 t^{(s-2m)} y^{(\gamma-2)} = 0. \quad (44)$$

The above is satisfied in five different cases

$$\begin{aligned}
s &= 0, \quad \gamma = 0, \\
s &= 0, \quad \gamma = 1 - 3\beta - \alpha, \\
\gamma &= 0, \quad s = 1 - 3l - m, \\
s &= 1 - 3l - m, \quad \gamma = 1 - 3\beta - \alpha, \\
m &= 1, \quad \alpha = 1, \quad s(s + 3l)C^2 + \epsilon\gamma(\gamma + 3\beta)A^2 = 0.
\end{aligned} \tag{45}$$

We now proceed to study these cases with some detail.

3.2.1 Case 1:

The selection $s = 0$, $\gamma = 0$ corresponds to the usual general relativity in $5D$ with $\phi = \text{constant}$. In this case the power-law cosmological solutions are well-known in the literature [46]. However, to make the paper self-consistent we provide them here.

From (27) we obtain $\alpha l + \beta(m - l) = 0$. Thus, assuming $l \neq m$ we get $\beta = \alpha l / (l - m)$. In this case the only power-law solution to the field equations is⁵

$$dS^2 = A^2 y^2 dt^2 - B^2 y^{2l/(l-1)} t^{2l} [dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)] - \frac{A^2 t^2}{(l-1)^2} dy^2. \tag{46}$$

(The field equations cannot be satisfied for $s = 0$, $\gamma = 0$ and $l = 1$, simultaneously). For $l = m$ the solution is

$$dS^2 = \frac{A^2}{y} dt^2 - B^2 y [dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)] + \epsilon dy^2. \tag{47}$$

Here, contrary to what we see in (46), the coordinate y is not restricted to be spacelike. Therefore, by virtue of extra symmetry discussed in [47], the transformation of coordinates $t = w^{3/2} \bar{y}$, $r = \bar{r} / \sqrt{w}$, $y = w \bar{t}$ with $w = \pm \sqrt{\epsilon}$ in (47) generates the “new” solution

$$dS^2 = d\bar{t}^2 - B^2 \bar{t} [d\bar{r}^2 + \bar{r}^2 (d\theta^2 + \sin^2 \theta d\varphi^2)] + \frac{\epsilon A^2}{\bar{t}} d\bar{y}^2. \tag{48}$$

We should note that although the last two line elements are diffeomorphic in $5D$ (allowing complex transformations of coordinates in $5D$), their interpretation in $4D$ is quite different⁶.

3.2.2 Case 2:

The choice $s = 0$ generates solutions where the scalar field ϕ depends only on y . Setting $s = 0$ and $\gamma = (1 - 3\beta - \alpha)$ in (44) we get $m = 3l(\alpha - \beta) / (\alpha - 1)$, where $\alpha \neq 1$. Now, (26) requires either $l = 1/2$ or $l = 0$.

The solution for $l = 1/2$ is

$$\begin{aligned}
dS^2 &= A^2 y^{(1+3\beta)/2} dt^2 - B^2 y^{2\beta} t [dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)] + \frac{\epsilon C^2}{t} dy^2, \\
\phi &= D y^{3(1-5\beta)/4}, \quad \beta = \frac{14 + 15\omega \pm 4\sqrt{6(4+3\omega)}}{94 + 75\omega}.
\end{aligned} \tag{49}$$

(When $\omega = -94/75 \approx -1.25$ the field equations require $\beta = 3/5$.)

⁵We exclude here static universes where $l = \beta = 0$. These admit both signatures ($\epsilon = \pm 1$) and require one of the following: ($\alpha = 1$, $m = 1$); ($\alpha = 0$, $m = 1$); ($\alpha = 1$, $m = 0$).

⁶As discussed in section 2, in this paper we follow the approach where our $4D$ spacetime is recovered by going onto a hypersurface $\Sigma_y : y = y_0 = \text{constant}$, although other techniques are also possible [48].

The solution for $l = 0$ is

$$\begin{aligned} dS^2 &= A^2 y^{2\alpha} dt^2 - B^2 y^{2\beta} [dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)] + \epsilon dy^2, \\ \phi &= Dy^{(1-\alpha-3\beta)}, \quad \omega = -\frac{12\beta^2 + 2(\alpha-1)(3\beta+\alpha)}{(1-\alpha-3\beta)^2}. \end{aligned} \quad (50)$$

It should be noted that this solution works perfectly well for $\alpha = 1$ provided $\omega = -4/3$. Also, when $\alpha = (1 - 3\beta)$ the denominator in ω vanishes and $\phi = \text{constant}$. What this suggests is that the choice $\alpha = (1 - 3\beta)$ should reproduce 5D general relativity. In fact, for this choice the field equations require either $(\alpha = -1/2, \beta = 1/2)$ or $(\alpha = 1, \beta = 0)$, which yield $\omega \rightarrow \infty$. For the former set of values the Brans-Dicke solution (50) reduces to (47), while for the latter it reduces to the third static solution mentioned in footnote 5.

3.2.3 Case 3:

The choice $\gamma = 0$ generates solutions where the scalar field ϕ depends only on t . Setting $\gamma = 0$ and $s = (l - 3l - m)$ in (44) we get $\alpha = -3\beta(l - m)/(m - 1)$, where $m \neq 1$. From (24) it follows that either $\beta = 1/2$ or $\beta = 0$. The solution corresponding to $\beta = 1/2$ is

$$\begin{aligned} dS^2 &= \frac{A^2}{y} dt^2 - B^2 y t^{2l} [dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)] + \epsilon C^2 t^{(1+3l)/2} dy^2, \\ \phi &= Dt^{3(1-5l)/4}, \quad l = \frac{14 + 15\omega \pm 4\sqrt{6(4+3\omega)}}{94 + 75\omega}. \end{aligned} \quad (51)$$

We note that this solution is related to (49) by the transformation $t \leftrightarrow y$, $\beta \leftrightarrow l$, $\epsilon C^2 \leftrightarrow A^2$.

The solution for $\beta = 0$ is

$$\begin{aligned} dS^2 &= dt^2 - B^2 t^{2l} [dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)] + \epsilon C^2 t^{2m} dy^2, \\ \phi &= Dt^{(1-m-3l)}, \quad \omega = -\frac{12l^2 + 2(m-1)(3l+m)}{(1-m-3l)^2}, \end{aligned} \quad (52)$$

which can formally be obtained from (50) after the transformation $t = (w^{1-\alpha}C/A)\bar{y}$, $r = w^{-2\beta}\bar{r}$, $y = w\bar{t}$ with $w = \pm\sqrt{\epsilon}$ and replacing α and β by m and l , respectively. In the limit where $m \rightarrow (1 - 3l)$, for which $\phi = \text{constant}$ and $\omega \rightarrow \infty$, we recover the general-relativistic solutions (48) and the second static metric mentioned in footnote 5.

It should be noted that (52) is the general power-law solution for the case where the metric depends on time alone. There is a very particular case, given below by (57) with $\beta = 0$, for which ϕ is allowed to depend on y .

3.2.4 Case 4

We now look for solutions to the field equations of the form

$$\begin{aligned} dS^2 &= A^2 y^{2(1-3\beta-\gamma)} dt^2 - B^2 y^{2\beta} t^{2l} [dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)] + \epsilon C^2 t^{2(1-3l-s)} dy^2, \\ \phi &= Dy^\gamma t^s. \end{aligned} \quad (53)$$

Substituting these expressions into (24) we get

$$\omega = -\frac{2(\gamma^2 - \gamma + 3\gamma\beta + 6\beta^2 - 3\beta)}{\gamma^2}, \quad \gamma \neq 0 \quad (54)$$

(If $\gamma = 0$ then we recover Case 3). Now from (27) we obtain

$$s = \frac{\gamma^2(1-6l) + 3\gamma[l + \beta(1-7l)]}{6\beta(1-2\beta) + \gamma(\gamma+1)}. \quad (55)$$

For these quantities we find that the choice $\beta = -\gamma/3$ ($s = -3l$) produces a simple particular solution, which is given below by (68).

The general solution to the field equations (24)-(28), in the case under consideration, is generated by a quadratic equation for l . The explicit expression for l is given by

$$l = \frac{9(6\beta + 7\gamma - 1)\beta^2 + 3(8\beta + \gamma - 2)\gamma^2 - 3(8\gamma + 3)\beta \pm \sqrt{3(3\beta + \gamma)(3 - 6\beta - \gamma)[6(2\beta - 1)\beta - (\gamma + 1)\gamma]}}{3(4\gamma^3 - 13\gamma^2 + 7\gamma - 3) + 9[11(6\beta + 5\gamma)\beta^2 - 15(3\beta - \gamma^2)\beta + 4(2 - 7\gamma)\beta]}. \quad (56)$$

Thus, (54)-(56) describe a two-parameter family of solutions. From a mathematical viewpoint, this expression might look cumbersome and/or non-interesting. However, from a physical viewpoint, the fact that l explicitly depends on β and γ means that the expansion rate of the universe, which is determined by l , is manifestly determined by the dynamics along the extra dimension.

The above solution can be simplified if we add some physical requirements. The simplest ones arise from the choice of the coordinate/reference system.

Synchronous reference system: The choice $g_{00} = 1$ is usual in cosmology; it corresponds to the so-called synchronous reference system where the coordinate t is the proper time at each point. Thus, setting $A = 1$ and $\alpha = (1 - 3\beta - \gamma) = 0$ in (53) the solution simplifies to

$$\begin{aligned} dS^2 &= dt^2 - B^2 y^{2\beta} t^{2l} [dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)] + \epsilon C^2 t^{2(1-3l-s)} dy^2, \\ \phi &= Dy^{(1-3\beta)} t^s, \quad \omega = \frac{6\beta(1-2\beta)}{(1-3\beta)^2}, \end{aligned} \quad (57)$$

where

$$\begin{aligned} l &= \frac{3(3\beta^2 - 1) \pm \sqrt{3(2-3\beta)[3\beta(\beta+1)-2]}}{3(3\beta^2 + 6\beta - 5)}, \\ s &= \frac{(1-3\beta)[3l(1+\beta)-1]}{3\beta(1+\beta)-2}. \end{aligned} \quad (58)$$

It is worthwhile to mention that there are no solutions having simultaneously $\alpha = 0$ and $\beta = 1/3$. Besides, the condition $\beta < 2/3$ ensures that l is a real number and $\omega > -4/3$. For completeness, we note that there is a unique solution for which the metric depends only on time, while the scalar field is a function of both t and y . Such a solution is generated by the choice $\beta = 0$ in the above expressions.

Gaussian normal coordinate system: A popular choice in the literature is to use the five degrees of freedom to set $g_{4\mu} = 0$ and $g_{44} = 1$. This is the so-called Gaussian normal coordinate system based on Σ_y . The solution in such coordinates can formally be obtained from (57)-(58) by changing $t \leftrightarrow y$, $l \leftrightarrow \beta$, $s \leftrightarrow \gamma$; setting $\epsilon C^2 = A^2$; and allowing the extra coordinate to have one or the other signature. The result is

$$\begin{aligned} dS^2 &= A^2 y^{2(1-3\beta-\gamma)} dt^2 - B^2 y^{2\beta} t^{2l} [dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)] + \epsilon dy^2, \\ \phi &= Dy^\gamma t^{(1-3l)}, \quad \omega = \frac{6l(1-2l)}{(1-3l)^2}, \end{aligned} \quad (59)$$

where

$$\begin{aligned} \beta &= \frac{3(3l^2 - 1) \pm \sqrt{3(2-3l)[3l(l+1)-2]}}{3(3l^2 + 6l - 5)}, \\ \gamma &= \frac{(1-3l)[3\beta(1+l)-1]}{3l(1+l)-2}. \end{aligned} \quad (60)$$

• Let us now go back to the general solution (53). For completeness we should consider the case where the denominator in (55) vanishes. In such a case, from (27) and (55) we find that the parameters β and γ are restricted to be

$$\begin{aligned}\gamma &= -3, \quad \beta = 1, \\ \gamma &= \frac{6l(5l-1)}{3l^2+6l-1}, \quad \beta = \frac{l(9l-1)}{3l^2+6l-1},\end{aligned}\tag{61}$$

The first set of parameters generates two families of solutions. Namely,

$$\begin{aligned}dS^2 &= A^2 y^2 dt^2 - B^2 y^2 t^{2l} [dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)] + \epsilon C^2 t^{2m_{(1,2)}} dy^2, \\ \phi &= Dy^{-3} t^{s_{(1,2)}}, \quad \omega = -\frac{4}{3},\end{aligned}\tag{62}$$

where $m_{(1)} = 1, s_{(1)} = -3l$ and $m_{(2)} = 3l-2, s_{(2)} = 3(1-2l)$. For the second set of parameters in (61), the expression for ω given by (54) simplifies to $\omega = (1-9l)/2l$ and the field equations have solutions only for two particular values of l . These are $l = 3/19$ and $l \approx 0.153$, which is the real root of the cubic equation $(246l^3 - 171l^2 + 40l - 3) = 0$.

3.2.5 Case 5

We now present the general solution for $\alpha = 1, m = 1$. We assume $s \neq 0, \gamma \neq 0$, which were already considered in the previous cases. Thus, from (45) we obtain⁷

$$C^2 = -\frac{\epsilon\gamma(\gamma+3\beta)}{s(s+3l)} A^2, \quad s \neq -3l.\tag{63}$$

From (27) we get

$$\beta = \frac{s[1-\gamma(1+\omega)]+3l+\gamma}{3(l-1)}, \quad l \neq 1.\tag{64}$$

Now (24) requires either

$$\omega = \omega_1 = \frac{s^2(1-\gamma) + s(3l+\gamma) - 3\gamma l(l-1)}{\gamma s^2},\tag{65}$$

or

$$\omega = \omega_2 = \frac{4s^2(1-\gamma) + 2s(1+3l)(3-\gamma) + 6l(l+1)(3+\gamma)}{s\gamma(4s+3+9l)}, \quad 4s+3-9l \neq 0.\tag{66}$$

Solution with $\omega = \omega_1$: Substituting (65) into (64) we obtain $\beta = l\gamma/s$. Consequently, from (63) it follows that $C^2 = -\epsilon(\gamma^2/s^2) A^2$. Now, it can be verified that the field equations are satisfied by

$$\begin{aligned}dS^2 &= A^2 y^2 dt^2 - B^2 y^{2\gamma l/s} t^{2l} [dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)] - \frac{\gamma^2}{s^2} A^2 t^2 dy^2, \\ \phi &= Dy^\gamma t^s, \quad \omega = \omega_1.\end{aligned}\tag{67}$$

The above solution works perfectly well for $l = 1$, although (64) requires $l \neq 1$. Besides the extra coordinate has to be spacelike ($\epsilon = -1$). In addition, for the choice $s = (l-1)\gamma$ the line element (67) becomes identical to the one in (46), but now $\phi = Dy^{s/(l-1)} t^s$ and $\omega = (l-s)/s$. With this choice, the 5D Brans-Dicke solution (67) goes over the 5D general-relativistic solution (46) in the limit⁸ $\omega \rightarrow \infty$ ($s \rightarrow 0$).

⁷This case was recently discussed in [49]. However, our solution (67) seems to be much simpler.

⁸To avoid misunderstandings, we should mention that the limit $\omega \rightarrow \infty$ does not necessarily implies that we recover general relativity. As an example consider the solution (49): For $\omega \rightarrow \infty$ we get $\beta = 1/5$ and $\phi = \text{constant}$, but the resulting metric does not satisfy the FE. A similar situation occurs in solutions (57)-(58) and (59)-(60) for $\beta \rightarrow 1/3$ and $l \rightarrow 1/3$, respectively.

For $s = -3l$ the solution is

$$\begin{aligned} dS^2 &= A^2 y^2 dt^2 - B^2 y^{-2\gamma/3} t^{2l} [dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)] + \epsilon C^2 t^2 dy^2, \\ \phi &= Dy^\gamma t^{-3l}, \quad \omega = -\frac{4}{3}. \end{aligned} \quad (68)$$

We note that this is the only solution to the FE, with $\alpha = m = 1$, which allows the extra coordinate to be timelike (see (46), (67)).

Solution with $\omega = \omega_2$: From the field equation (26) we obtain the condition

$$(s + 3l)^2 [\gamma(3l + 1) + s(\gamma - 1)] = 0. \quad (69)$$

Since $s + 3l \neq 0$ (see (63) and (68)) it follows that

$$\begin{aligned} s &= -\frac{\gamma(3l + 1)}{\gamma - 1}, \quad \gamma \neq 1 \\ \gamma &= 1, \quad l = -\frac{1}{3}. \end{aligned} \quad (70)$$

It can be verified that when $\omega = \omega_2$ the solution to the field equations is given by (67) with the choice of parameters given above. When the denominator in (66) vanishes, the solution is (67) with

$$s = -\frac{3(1 + 3l)}{4}. \quad (71)$$

Thus, the choice $\omega = \omega_2$ does not generate new solutions to the FE. It just singles out some particular values for the parameters. Consequently, (67) is the general power-law solution with $\alpha = 1$, $m = 1$.

4 Reduced Brans-Dicke cosmology in 4D

We now proceed to study the effective 4D picture generated by the 5D solutions discussed in the preceding section. For the line element (23), the cosmological metric induced on spacetime hypersurfaces $\Sigma_y : y = y_0 = \text{constant}$ is just the spacetime part of (23), and the non-vanishing components of the induced Brans-Dicke energy-momentum tensor (15) are

$$\begin{aligned} \frac{8\pi}{\phi} T_0^{(BD)} &= \frac{1}{n^2} \left(\frac{\ddot{\Phi}}{\Phi} - \frac{\dot{n}\dot{\Phi}}{n\Phi} \right) + \frac{\epsilon}{\Phi^2} \left[\frac{n^{**}}{n} - 3 \frac{a^*}{a} \left(\frac{a^*}{a} + \frac{\phi^*}{\phi} \right) - \frac{n^* \Phi^*}{n\Phi} + \frac{\omega \phi^{*2}}{2\phi^2} \right] + \frac{V}{2\phi}, \\ \frac{8\pi}{\phi} T_1^{(BD)} &= \frac{\dot{a}\dot{\Phi}}{n^2 a \Phi} + \frac{\epsilon}{\Phi^2} \left[\frac{a^{**}}{a} - \frac{a^*}{a} \left(\frac{a^*}{a} + \frac{\Phi^*}{\Phi} + \frac{2n^*}{n} + \frac{2\phi^*}{\phi} \right) - \frac{\phi^*}{\phi} \left(\frac{n^*}{n} - \frac{\omega \phi^*}{2\phi} \right) \right] + \frac{V}{2\phi}, \end{aligned} \quad (72)$$

where $V = V(\phi)$ should be determined from (19). We note that $T_2^{(BD)} = T_3^{(BD)} = T_1^{(BD)}$, which means that the induced EMT looks like a perfect fluid with energy density $\rho = T_0^{(BD)}$ and isotropic pressure $p = -T_1^{(BD)}$.

Let us first calculate the potential. With this aim we substitute the power-law line element (43) into (19) and evaluate on Σ_y . We obtain

$$\left[\phi \frac{dV}{d\phi} \right]_{|\Sigma_y} = -2ms(\omega + 1) D^{2/s} A^{-2} y_0^{2(\gamma - \alpha)s} \phi^{(s-2)/s} - \epsilon C D^{2m/s} C^{-2} y_0^{2(\gamma m - s)/s} \phi^{(s-2m)/s}, \quad s \neq 0, \quad (73)$$

where

$$\mathcal{C} = \gamma\omega [3\gamma + 2(\alpha + 3\beta - 1)] + 2[\gamma^2 - \gamma - 3\beta(\alpha + \beta)]. \quad (74)$$

• It turns out that \mathcal{C} vanishes identically for all solutions discussed in Cases 3 and 4. Therefore in these cases, integrating (73) and setting the constant of integration equal to zero we find

$$V(\phi) = \begin{cases} -\frac{2ms^2(\omega+1)D^{2/s}y_0^{2(\gamma-\alpha s)/s}}{(s-2)A^2}\phi^{(s-2)/s}, & s \neq 0, \quad s \neq 2, \\ -4m(\omega+1)DA^{-2}y_0^{(\gamma-2\alpha)}\ln\phi, & s = 2. \end{cases} \quad (75)$$

Thus, the effective potential vanishes either in the particular case where $\omega = -1$ or in the Gaussian normal coordinate system ($m = 0$), i.e., for the solution (59)-(60).

• In Case 5, $\mathcal{C} \neq 0$ and from (73) we get

$$V(\phi) = \begin{cases} \frac{s[\omega s(s-2+6l)+6l(1-2l)]D^{2/s}y_0^{2(\gamma-s)/s}}{(s-2)A^2}\phi^{(s-2)/s}, & s \neq 0, \quad s \neq 2, \\ 6l(2\omega+1-2l)DA^{-2}y_0^{(\gamma-2\alpha)}\ln\phi, & s = 2. \end{cases} \quad (76)$$

• The parameter s is equal to zero only in Cases 1 and 2. In Case 1, the general-relativistic 5D solution (46) reduces on every hypersurface Σ_y to the spatially flat FRW models of general relativity, viz.,

$$ds^2 = dS_{\Sigma_y}^2 = A^2 y_0^2 dt^2 - B^2 y_0^{2l/(l-1)} t^{2l} [dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)], \quad (77)$$

with

$$\rho = \rho_0 \left(\frac{t_0}{t} \right)^2, \quad p = n\rho, \quad (78)$$

where $\rho_0 = 3l^2 / (8\pi t_0^2 A^2 y_0^2)$ is the energy density measured at some time $t = t_0$ and

$$n = \frac{2-3l}{3l}. \quad (79)$$

• In Case 2, to evaluate V we substitute (49) into (19). We find that $\phi dV/d\phi$ vanishes identically by virtue of the equation relating β and ω . The same is true for solution (50). Therefore, without loss of generality we can set $V = 0$. Now, from (72) we find that (49) reduces on Σ_y to a radiation dominated universe ($\rho = 3p$), regardless of ω . Similarly, (50) reduces to empty space ($\rho = p = 0$) on Σ_y .

4.1 4D cosmologies induced by the 5D solutions of Cases 3 and 4

Cases 3 and 4, with $s \neq 2$, give more general BD cosmological models⁹ in 4D. In fact, from (72), using (75) we get

$$\begin{aligned} \frac{8\pi}{\phi} T_0^{0(BD)} &= \frac{m[(m-1)(s-2) - s^2(\omega+1)]}{(s-2)y_0^{2\alpha} A^2 t^2}, \\ T_1^{1(BD)} &= -n T_0^{0(BD)}, \end{aligned} \quad (80)$$

where

$$n = \frac{s^2(\omega+1) - l(s-2)}{(m-1)(s-2) - s^2(\omega+1)}. \quad (81)$$

Thus, the effective pressure and density satisfy the barotropic equation of state

$$p = n\rho. \quad (82)$$

⁹If $s = 2$ and $m \neq 0$, the logarithmic potential leads to a violation of the weak energy condition $\rho \geq 0$. Therefore, to avoid such a violation we should set $m = 0$ when $s = 2$, which in turns implies that $T_{\mu\nu}^{(BD)} = 0$ on every Σ_y .

We immediately notice that the $5D$ solutions (59)-(60) in the Gaussian normal frame ($m = 0$) reduce to scalar-vacuum cosmologies in $4D$. One can verify that for all solutions in Cases 3 and 4, n can be written as

$$n = \frac{2 - 3l - s}{3l}, \quad (83)$$

and

$$m = 1 - 3l - s = -1 + 3ln. \quad (84)$$

We can use these expressions to eliminate s and m in (80)-(81), viz.,

$$\begin{aligned} \omega &= \frac{6l(2 + 3n) - 6l^2[3n(n + 1) + 2] - 4}{[3l(n + 1) - 2]^2}, \\ \frac{8\pi}{\phi} T_0^{0(BD)} &= \frac{(l + 2 - 3ln)(1 - 3ln)}{(n + 1)A^2 y_0^{2\alpha} t^2}, \quad T_1^{1(BD)} = -nT_0^{0(BD)}. \end{aligned} \quad (85)$$

As a consequence, we can formulate the reduced Brans-Dicke cosmological models in $4D$, derived from the $5D$ solutions of Cases 3 and 4, solely in terms of l and n as

$$\begin{aligned} ds^2 &\equiv dS_{|\Sigma_y}^2 = A^2 y_0^{2\alpha} dt^2 - B^2 y_0^{2\beta} t^{2l} [dr^2 + r^2 (d\theta^2 + \sin\theta d\varphi^2)], \\ \rho &= \rho_0 \left(\frac{t_0}{t} \right)^{3l(n+1)}, \\ \phi &= \phi_0 \left(\frac{t}{t_0} \right)^{[2-3l(n+1)]}, \\ V &= V_0 \left(\frac{\phi}{\phi_0} \right)^{3l(n+1)/[3l(n+1)-2]}, \end{aligned} \quad (86)$$

with

$$\begin{aligned} \phi_0 &= \frac{8\pi(n+1)\rho_0 \bar{t}_0^2}{(l+2-3ln)(1-3ln)}, \quad \bar{t}_0 \equiv Ay_0^\alpha t_0 \\ V_0 &= \frac{16\pi\rho_0 [l(1+3n^2) - 2n]}{2+l(1-3n)}, \end{aligned} \quad (87)$$

where ρ_0 refers to the value of the energy density at some arbitrary fixed time t_0 , and

$$l = l_{(\pm)} = \frac{6(\omega + 1) + 3n(2\omega + 3) \pm \sqrt{9n^2 + 12(1 + \omega)(3n - 1)}}{9(\omega + 2)n^2 + 18(\omega + 1)n + 3(3\omega + 4)}. \quad (88)$$

We recall that, in principle, l is an observable quantity because it is related to the acceleration parameter $q = -a\ddot{a}/\dot{a}^2$ by

$$l = \frac{1}{q + 1}.$$

The denominator of ω in (85) vanishes when $l = 2/3(n + 1)$ and $\omega \rightarrow \text{sign}(3n - 1) \times \infty$. In this limit $s = 0$, $\phi = \text{constant}$ and from (84) it follows that $m = (n - 1)/(n + 1)$, which means that the extra dimension contracts, or compactifies ($m < 0$), for any $-1 < n < 1$. Consequently, formally for $|\omega| \rightarrow \infty$ we recover the usual spatially flat FRW cosmology of ordinary general relativity.

Matter-dominated universe: Since the present epoch of the universe is matter dominated let us consider the case where $n = 0$. In this case the weak energy condition is satisfied for any $l > 0$. On the other hand, the condition $\omega > -3/2$ restricts the range of l to be either $l < 2 \left(1 - \sqrt{2/3}\right) \approx 0.37$ or $l > 2 \left(1 + \sqrt{2/3}\right) \approx 3.633$. The former range requires $q > (1 + \sqrt{6})/2 \approx 1.72$, which is inapplicable to the present epoch. However, the latter range leads to $q < (1 - \sqrt{6})/2 \approx -0.72$, which is within the current observed measurements $q = -0.67 \pm 0.25$ [50]. We note that the parameter l is a real number only if $(1 + \omega) \leq 0$.

Radiation-dominated universe: For $n = 1/3$ we find that $T_0^{0(BD)} > 0$ in (85) requires $l < 1$, i.e., $q > 0$. Now from (88) we get $l_{(+)} = 1/2$ and $l_{(-)} = 2(1 + \omega)/(4\omega + 5)$. The solution $l_{(+)} = 1/2$ is identical to the radiation-dominated epoch of general relativity, regardless of the specific value of ω . The second solution violates the condition $0 < l < 1$ for any $\omega < -1$, as required by the dust model.

Therefore, the effective BD cosmology in $4D$ (86)-(88) gives a decelerated radiation era, which is consistent with the big-bang nucleosynthesis scenario of general relativity ($l = 1/2$), as well as an accelerating matter dominated era compatible with present observations.

To avoid misunderstanding, we should mention that the above discussion assumes that the parameters n and ω are independent, which is true in the $4D$ cosmologies generated by (52) and (53): giving n and ω we obtain l from (88). This in turn defines the rest of the solution (86)-(87). The $5D$ solutions (51) and (57) only have one independent parameter. Therefore they lead to a more restricted class of BD cosmologies in $4D$. In particular, one can show that (51) yields shrinking (instead of expanding) dust models. Regarding the solutions in the synchronous frame (57), they produce $4D$ models of expanding dust, but do not yield accelerated expansion.

• If we assume $V = 0$ at the beginning, then we obtain a set of BD cosmologies in $4D$ which precludes the existence of a dust-filled universe. In fact, in the case where $V = 0$, ($m \neq 0, \omega = -1$) instead of (80) we have

$$\begin{aligned} \frac{8\pi}{\phi} T_0^{0(BD)} &= \frac{m(m-1)}{t^2 (Ay_0^\alpha)^2}, \\ \frac{8\pi}{\phi} T_1^{1(BD)} &= \frac{lm}{t^2 (Ay_0^\alpha)^2}. \end{aligned} \quad (89)$$

Thus, a dynamical universe ($l \neq 0$) with $p = 0$ requires $\rho = 0$.

4.2 $4D$ cosmologies induced by the $5D$ solutions of Case 5

Finally, let us study the effective $4D$ world generated by the $5D$ solutions discussed in Case 5. We emphasize that these are the only solutions demanding the extra dimension to be spacelike.

Now the effective potential is given by (76). To avoid contradictions with the weak energy condition we disregard the logarithmic potential (see footnote 9). In this context, from (72) we find the effective EMT as

$$\frac{8\pi}{\phi} T_0^{0(BD)} = \frac{3l[s(s+3l)] + \gamma l(1-s-3l)}{y_0^2 t^2 \gamma (s-2) A^2}, \quad T_1^{1(BD)} = -n T_0^{0(BD)}, \quad (90)$$

where n is the same as in Cases 3 and 4, namely

$$n = \frac{2 - 3l - s}{3l}.$$

As a consequence, from (65) we find that γ can be expressed as

$$\gamma = \frac{(3ln - 2)[3l(n+1) - 2]}{3l(n+1)[3l(n+1)(1+\omega) - 4\omega - 3] + 3l(l-1) + 2(1+2\omega)}. \quad (91)$$

Substituting this into (90) we get

$$\frac{8\pi}{\phi} T_0^{0(BD)} = \frac{2(2l-1-2\omega) + 3l(n+1)[4\omega+3-l-3l(n+1)(\omega+1)]}{y_0^2 t^2 (n+1) A^2}. \quad (92)$$

After some calculations we find that the effective cosmology in 4D is formally identical to (86) except for the fact that now¹⁰

$$\begin{aligned} \phi_0 &= \frac{8\pi(n+1)\rho_0 \bar{t}_0^2}{2(2l-1-2\omega) + 3l(n+1)[4\omega+3-l-3l(n+1)(\omega+1)]}, \quad \bar{t}_0 \equiv Ay_0 t_0 \\ V_0 &= \frac{8\pi\rho_0[3l(n+1)-2]\{2[1+\omega(1-n)] + l[3\omega(n^2-1)-4]\}}{2(2l-1-2\omega) + 3l(n+1)[4\omega+3-l-3l(n+1)(\omega+1)]}. \end{aligned} \quad (93)$$

Let us consider the above solution for large values of ω , in accordance with the solar system bound on $\omega(> 600)$. From the above expressions it follows that $\rho > 0$, $\phi > 0$ require l in the range

$$l = \frac{2}{3(n+1)} \pm \frac{2}{3\sqrt{3}(n+1)^{3/2}\sqrt{\omega}} + O\left(\frac{1}{\omega}\right).$$

Consequently, as $\omega \rightarrow \infty$ we recover the usual spatially flat FRW cosmologies of general relativity.

Let us now consider a matter-dominated universe ($n = 0$). Using that $l = 1/(q+1)$ we find that the positivity of the effective density requires

$$\omega < \frac{9q-2q^2-1}{(2q-1)^2}.$$

For the sake of argument, let us consider an accelerated expansion with $q = -2/3$, which corresponds to $l = 3$. This requires $\omega < -71/49 \approx -1.449$. In general, $q < 0$ requires $\omega < -1$. As $q \rightarrow 1/2$, which is the general-relativistic value of q for dust, the theory becomes consistent with increasingly larger values of ω .

We note that the solution can give accelerating matter-dominated era as well as decelerating radiation-dominated era ($n = 1/3$) for the same value of ω in the range $-3/2 \leq \omega < -1$. For example if we choose $\omega = -1.45$, then a consistent model of decelerating radiation era is obtained for $l < 0.88$, and accelerated dust era for $l > 3$, approximately.

• For completeness, let us consider the case where $V = 0$. Coming back to the original expression for $\phi dV/d\phi$, we find that $V = 0$ requires

$$\gamma = \frac{s(s+6l-2)(s+3l)}{(s+3l-1)[s^2+s(3l-2)+6l(l-1)]}. \quad (94)$$

Substituting into (65) we obtain

$$\omega = \frac{6(2l-1)l}{s(s+6l-2)}. \quad (95)$$

Thus, for $V = 0$ (omitting intermediate calculations) we find

$$p = n\rho, \quad n = \frac{2-s-3l}{3l}, \quad \omega = \frac{2(2l-1)}{(1-n)[2-3l(n+1)]}, \quad \frac{8\pi}{\phi} T_0^{0(BD)} = \frac{2-l+3nl[l(3n+1)-3]}{y^2 A^2 t^2 (1-n)}. \quad (96)$$

Consequently, the effective BD cosmology in 4D can be written as

$$\begin{aligned} \phi &= \phi_0 \left(\frac{t}{t_0} \right)^{2(1-3n)/[4+3\omega(1-n^2)]}, \\ a &= a_0 \left(\frac{t}{t_0} \right)^{2[1+\omega(1-n)]/[4+3\omega(1-n^2)]}, \end{aligned} \quad (97)$$

¹⁰Here ϕ_0 and V_0 depend on three independent parameters, n , l and ω . If one fixes ω as in (85), then (93) reduces to (87).

with

$$\phi_0 = \frac{4\pi\rho_0\bar{t}_0^2 [4 + 3\omega(1 - n^2)]^2}{(3 + 2\omega) [4 - 6n + 3\omega(n - 1)^2]}, \quad \bar{t}_0 \equiv Ay_0 t_0. \quad (98)$$

In this parameterization, the $n = 0$ models become identical to those presented in the original BD paper [38]. For $n \neq 0$ we recover the type A-I solutions discussed in [37], although in a slightly different notation. From (96) it is easy to see that any ω in the range $-2 < \omega < -3/2$ ($1 < l < 2$) yields a matter-dominated universe with accelerated expansion. However, this range is inconsistent with a radiation dominated epoch, which demands $\omega = -3/2$ ($l = 1/2$).

5 Summary

Brans-Dicke theory in $4D$ can explain the observed accelerated expansion of the present matter-dominated universe, without invoking the presence of dark energy, if ω is allowed to take some value in the range $-2 < \omega < -3/2$. However, ω in this range does not produce a consistent radiation-dominated epoch with decelerating expansion, as requires the big-bang nucleosynthesis scenario. One way out of this problem is to introduce a self interacting potential [31]. Another way is to consider a modified BD theory with a varying ω [32]. Both approaches are not free of criticism. In one of them the potential is added by hand, while the other creates the necessity of finding the fundamental mechanism driving the variation of ω .

A third way out of this problem is to resort to higher dimensions. In [40] it was shown that the BD field equations in $5D$ can be reduced, on a hypersurface orthogonal to the extra dimension, to those of GR in $4D$ coupled to two scalar fields. These fields may account for the late-time accelerated expansion provided several other conditions are met: the extra coordinate is cyclic; $T_{AB}^{(5)} \neq 0$; $T_{AB}^{(5)} \hat{n}^A = 0$; $a^3 \Phi^m = \text{constant}$, where m is a positive constant; the effective matter in $4D$ has negligible pressure.

In this work we have presented an approach which is free of the above criticism and conditions. Without making any assumption on the nature of the extra coordinate or the matter content in $5D$, we have shown that the BD field equations in $5D$ are equivalent to those of BD in $4D$ derived from the action (20) with non-vanishing scalar potential $V = V(\phi)$. The potential is not introduced by hand, instead the reduction procedure provides an expression, namely (19), that determines the shape of $V(\phi)$ up to a constant of integration [(75)-(76)]. It also establishes the explicit formulae for the effective EMT in $4D$ [(14)-(16)]. This extends and generalize some previous results recently obtained by the present author [53].

In the context of FRW cosmological models we have integrated the vacuum FE in $5D$ under the sole assumption of separation of variables. We analyzed in detail the class of power-law solutions (43). We obtained a large family of solutions, namely (67), that has three free parameters; various families with two free parameters [(50), (52), (53)]; one free parameter [(49), (51), (57), (59)], and solutions that only exist for $\omega = -4/3$ [(62), (68)]. Thus, the spectrum of BD power-law solutions in $5D$ is significantly larger than the one of GR in $5D$, which can have at most one free parameter (Case 1).

We discussed the effective $4D$ world generated by our solutions. We found that the theory yields power-law and logarithmic potentials, except for the solutions derived in Case 2 and those in the Gaussian normal frame where we can set $V = 0$, without loss of generality. Certainly, one can assume $V = 0$ at the outset in (19), in which case the theory in $4D$ would be identical to the original BD [38]. However, such assumption imposes a strong constraint on the parameters of the solutions leading to restricted cosmological models in $4D$ which have no, or little, relevance to the problem of cosmic acceleration [(89), (96)].

We found that all BD models in $4D$ can formally be expressed as in (86) with ϕ_0 and V_0 given by (87) or (93). The models are given in terms of the parameters n , l and ω . When at least two of them are independent, which occurs for a large number of solutions [(50), (52), (53), (67)] we can obtain models that for the same ω can give the present accelerated expansion and a decelerated radiation-dominated epoch as required by primordial nucleosynthesis.

We should mention that as in the conventional $4D$ Brans-Dicke theory, in our models the (effective) EMT obeys the ordinary conservation law $D_\mu T_\nu^{\mu(BD)} = 0$ (the same as in Einstein's theory), which in the cosmological realm yields the usual equation of motion

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0. \quad (99)$$

Although, we do not require the introduction of a self interacting potential in $4D$ or matter in $5D$, the range of ω obtained in our work is consistent with the one obtained by a number of authors in the literature in the context of several cosmological models and different versions of BD theory in $4D$ and $5D$ [31]-[41]: In all cases cosmic accelerated expansion requires $\omega < -1$.

There is also evidence in the literature that small values of $|\omega|$ are required by inflationary models [51] and structure formation [52] in scalar-tensor theory. Certainly, these theoretical results contradict solar system experiments which impose the constraint $\omega > 600$. However, this solar system limit does not necessarily imply that the evolution of the universe is, at all scales, close to general relativity. The fact is that GR is poorly tested on cosmic scales [29] and no experiment has been done to test BD in cosmological scale yet. Consequently, one cannot discard scalar-tensor scenarios of the sort discussed here and in [31]-[41], [51]-[52] on the basis that in cosmological scales ω does not meet solar system requirements.

Thus, the problem of (in)compatibility between astronomical and cosmological requirements remains open. Our recent work [53] indicates that a variable (homogeneous) ω cannot solve the problem. Perhaps, local inhomogeneities might give rise to high values of ω consistent with astronomical observations [32].

Appendix A: Integrating the field equations

In section 3 we have seen that the scalar-vacuum Brans-Dicke FE for the cosmological metric (23) give five partial differential equations to be solved for four unknowns, $n(t, y)$, $a(t, y)$, $\Phi(t, y)$ and $\phi(t, y)$. The assumption that the metric is separable increases the number of unknowns to six. The aim of this appendix is to show, by means of explicit integration, the consistency of this assumption with the field equations.

I: First, we assume that $c_1 \neq 0$. Then from (33) we get

$$Q = Q_0 F^{-(c_2/3c_1)} W^{-(c_3/3c_1)}, \quad (\text{A-1})$$

where Q_0 is a constant. Substituting this and (34) into the wave equation (28) we obtain

$$c_1 F W \ddot{W} - \dot{W} \left[c_3 F \dot{W} + (c_2 - c_1) W \dot{F} \right] + \frac{\epsilon c_1 c_2 C_s N_0^2 [c_2 + 3(c_1 - c_3)]}{(3\omega + 4)} F^{-1} W^2 = 0. \quad (\text{A-2})$$

When $c_1 \neq 0$ the assumption of separability requires $k = 0$ and the fulfillment of one of the following conditions (i) $C = 0$, (ii) $\eta_0 = 0$, (iii) $c_2 = 0$ (cases Ia and Ib below).

Ia: Now we substitute the metric functions (29), including (32), (34) and (A-1), into the FE (24)-(26) with $k = 0$ and either $C = 0$ or $\eta_0 = 0$. We find that (24) is a second-order differential equation for W , while (25) and (26) contain the second derivatives of W and F . Next, we isolate \ddot{W} from (A-2) and substitute it into (24) to obtain the following first-order differential equation

$$\frac{\hat{c}_1 \dot{W}^2}{W^2} + \frac{\hat{c}_2 \dot{W} \dot{F}}{W F} + \frac{\hat{c}_3 \dot{F}^2}{F^2} + \frac{\hat{c}_4}{F^2} = 0 \quad (\text{A-3})$$

with

$$\begin{aligned} \hat{c}_1 &\equiv 3c_1^2 \omega + 6c_1 c_3 - 2c_3^2, \\ \hat{c}_2 &\equiv 6c_1 (c_2 + c_3 - c_1) - 4c_2 c_3, \\ \hat{c}_3 &\equiv 2c_2 (3c_1 - c_2), \\ \hat{c}_4 &\equiv 3\epsilon c_1^2 C_s N_0^2 [\eta_0 / (3\omega + 4)] - 2c_2^2. \end{aligned} \quad (\text{A-4})$$

A similar substitution into (26) yields an expression for \ddot{F} , whose explicit form we omit, as $\ddot{F} = \ddot{F}(W, F, \dot{W}, \dot{F})$. With these \ddot{W} and \ddot{F} we find that (25) reduces to (A-3). As a result there are two independent equations, viz., (A-2) and (A-3), for the two unknowns W and F .

Ib: For $c_1 \neq 0$, $c_2 = 0$ ($C \neq 0, \eta_0 \neq 0$), equation (A-2) can be easily integrated. The general solution is $F = F_0 \dot{W}^{-1} W^{c_3/c_1}$, where F_0 is a constant of integration. Using this expression, and (A-1) with $c_2 = 0$, we find that both (24) and (26) generate the same second-order differential equation for W , namely,

$$c_1 (c_1 - c_3) W \ddot{W} + \frac{\dot{W}^2}{2} \left[c_1^2 \omega + \frac{4}{3} c_3^2 - \frac{3\epsilon c_1^2 (c_1 - c_3)^2 (C_s + C) N_0^2}{F_0^2 (3\omega + 4)} W^{2(c_1 - c_3)/c_1} \right] = 0. \quad (\text{A-5})$$

On the other hand, (25) becomes a third-order differential equation for W , which upon substitution of (A-5) into it reduces to the identity $0 = 0$.

• A particular solution to (A-2), for $c_2 = 0$, is $W = \text{constant}$, which leads to $Q = \text{constant}$. In turn, (24) and (26) become identical to each other, both requiring $c_3 = c_1$, in agreement with (A-5). With this requirement (25) yields $F \propto t$.

II: We now consider the case where $c_1 = 0$ and $k \neq 0$ with $c_2 \neq 0$ and $c_3 = 0$ (cases IIa and IIb bellow).

IIa: First we assume $c_2 \neq 0$, in which case from (33) we get $F = F_0 W^{-c_3/c_2}$. Substituting this and (34) into the wave equation (28) we obtain¹¹

$$c_2 Q W \ddot{W} - \dot{W} (c_3 Q \dot{W} - 3c_2 W \dot{Q}) + \frac{\epsilon c_2^2 C_s N_0^2 (c_2 - 3c_3)}{F_0^2 (3\omega + 4)} Q W^{2(c_2 + c_3)/c_2} = 0. \quad (\text{A-6})$$

When $c_2 \neq 0$ separability (36) requires $\eta_0 C = 0$. Following the same steps as above, we obtain a first-order differential equation by substituting \ddot{W} from (A-6) into (24). Namely,

$$(c_2 \omega + 2c_3) \frac{\dot{W}^2}{W^2} + 6(c_3 - c_2) \frac{\dot{W} \dot{Q}}{W Q} - 6c_2 \frac{\dot{Q}^2}{Q^2} - \frac{6c_2 N_0^2}{P_0^2} \frac{k}{Q^2} - \frac{\epsilon c_2 C_s N_0^2}{F_0^2 (3\omega + 4)} [c_2^2 (2\omega + 3) - 2c_2 c_3 + 3c_3^2] W^{2c_3/c_2} = 0. \quad (\text{A-7})$$

Taking the time derivative of (A-7) and replacing into it the expression for \ddot{W} obtained from (A-6), after some lengthy algebraic manipulations, we derive (26), which now is a second-order differential equation for Q . On the other hand, (25) reduces to (A-7) after we substitute \ddot{W} and \ddot{Q} into it. Thus, (A-6)-(A-7) generate all the separable solutions with $c_1 = 0$, $k \neq 0$ and $c_2 \neq 0$.

IIb: We now let $c_2 = 0$ and assume $c_3 \neq 0$. In this case one can generate solutions with $k \neq 0$ and $\eta_0 C \neq 0$. From (33) it follows that $W = \text{constant}$. Consequently, the wave equation (28) is satisfied identically. From (24) and (26) we obtain the differential equations that govern F and Q . Namely,

$$\frac{\dot{Q}^2}{Q^2} + \frac{\dot{F} \dot{Q}}{F Q} + \left(\frac{N_0}{P_0} \right)^2 \frac{k}{Q^2} + \frac{\epsilon c_3^2 (C_s + C) N_0^2}{2 (3\omega + 4) F^2} = 0, \quad (\text{A-8})$$

and

$$Q \ddot{Q} + \dot{Q}^2 + k \left(\frac{N_0}{P_0} \right)^2 - \frac{\epsilon c_3^2 (C_s + C) N_0^2 Q^2}{2 (3\omega + 4) F^2} = 0. \quad (\text{A-9})$$

Taking the time derivative of (A-8) and using (A-9) we obtain an expression for \ddot{F} , which we omit here, as $\ddot{F} = \ddot{Q}(F, Q, \dot{F}, \dot{W})$. Once again after some manipulations, one can verify that (25) reduces to (A-8) after \ddot{F} and \ddot{Q} are substituted into it.

Thus, we have obtained the differential equations that generate all distinct cases of separable solutions. They demonstrate, in a constructive way, that the FE are consistent with the assumption of separability. In the case where $c_1 = c_2 = c_3 = 0$ the metric functions become independent of the extra coordinate. For this case, the solutions to the FE as well as their cosmological applications have recently been discussed by the present author in [53].

¹¹In general, when $c_2 \neq 0$ from (33) we get $F = F_0 W^{-c_3/c_2} Q^{-3c_1/c_2}$ and the wave equation (28) yields

$$c_2 Q W \ddot{W} - \dot{W} [c_3 Q \dot{W} + 3(c_1 - c_2) W \dot{Q}] + \frac{\epsilon c_2^2 C_s N_0^2 [c_2 + 3(c_1 - c_3)]}{F_0^2 (3\omega + 4)} Q^{(6c_1 + c_2)/c_2} W^{2(c_2 + c_3)/c_2} = 0.$$

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